Accuracy problem and a possible solution are described in:

On High-Order Fluctuation-Splitting Schemes for Navier-Stokes Equations, Computational Fluid Dynamics 2004: Proceedings of the Third International Conference on Computational Fluid Dynamics, ICCFD, Toronto, 12-16 July 2004, Springer, 2006.

On the Accuracy of Fluctuation-Splitting Schemes for Advection-Diffusion Equation

Hiroaki Nishikawa

December 2003

1 2nd-Order LDA-Galerkin Scheme

We consider high-order spatial discretization of the following model equation over a triangular grid.

$$\partial_t u + a u_x + b u_y = \mu \, \mathrm{div}(\mathrm{grad} u). \tag{1}$$

2 Truncation Error for Viscous Part

2.1 Standard Galerkin (STD Galerkin)

The standard Galerkin

$$S_j \frac{du_j}{dt} = -\frac{\mu}{2} \sum_{T \in \{T_j\}} (\mathrm{grad} u)^T \cdot \mathbf{n}_T \tag{2}$$

where S_j is the area of the median dual cell and $\mathbf{n}_T$ is the scaled inward normal for the triangle T of the edge opposite to the node j.

On a regular triangular grid, we expand a solution and insert it into the update formula to find the truncation error. At convergence, we have

$$\begin{aligned} 0 &= (u_{xx} + u_{yy}) + \frac{h^2}{12} \left[u_{xxxx} + u_{yyyy} \right] \\ & \quad + \frac{h^4}{360} \left[u_{xxxxxx} + u_{yyyyyy} \right]. \end{aligned} \tag{3}$$

The error is $\mathcal{O}(h^2)$. Also, note that the error depends on the 4th (and higher) derivatives of the solution, meaning the scheme is exact for polynomial solutions of degree 3.

2.2 High-Order Discretization: Gradient Reconstruction (GR)

$$u_j^{n+1} = u_j^n - \omega \frac{\mu}{2} \sum_{T \in \{T_j\}} (\text{grad}u)_{high}^T \cdot \mathbf{n}_T \tag{4}$$

On a regular grid, the resulting scheme has the following discretization error.

$$\begin{aligned} 0 &= (\Delta u) + \frac{h^2}{12}\left[\partial_{xx} + \partial_{xy} + \partial_{yy}\right](\Delta u) \\ &+ h^4 \left[\frac{1}{360}\partial_{xxxx} + \frac{1}{48}\partial_{xxxy} + \frac{43}{720}\partial_{xxyy} + \frac{1}{48}\partial_{xyyy} + \frac{1}{360}\partial_{yyyy}\right](\Delta u) \\ &+ \frac{h^4}{24} u_{xxxyyy} \end{aligned} \tag{5}$$

where $\Delta u = u_{xx} + u_{yy}$.

2.3 Gradient-Based Discretization (GBD) of Laplacian

Defining the viscous fluctuation as

$$\phi_v^T = \mu \sum_{i \in \{i_T\}} \frac{1}{2} (\text{grad}u)_i \cdot \mathbf{n}_i, \tag{6}$$

we distribute it with the equal weight 1/3,

$$S_j u_j^{n+1} = S_j u_j^n - \sum_{T \in \{T_j\}} \frac{1}{3} \phi_v^T \tag{7}$$

or equivalently,

$$S_j u_j^{n+1} = S_j u_j^n - \mu \sum_{T \in \{T_j\}} \left(\overline{u_x}_T, \overline{u_y}_T\right) \cdot \mathbf{n}_T \tag{8}$$

On a regular grid, inserting the expanded solution into the update, we obtain

$$\begin{aligned} 0 &= (\Delta u) + \frac{h^2}{3}\left[\partial_{xx} + \partial_{xy} + \partial_{yy}\right](\Delta u) \\ &+ h^4 \left[\frac{2}{45}\partial_{xxxx} + \frac{1}{12}\partial_{xxxy} + \frac{3}{20}\partial_{xxyy} + \frac{1}{12}\partial_{xyyy} + \frac{2}{45}\partial_{yyyy}\right](\Delta u) \\ &+ \frac{h^4}{18} u_{xxxyyy}. \end{aligned} \tag{9}$$

2.4 SUPG for Laplacian

Viscous fluctuation is the same as in the previous scheme, but distributed by the SUPG scheme whose distribution coefficient is given by

$$\beta = \frac{1}{3}\left(1+\tau\frac{k_i}{\sum \max(0,k_i)}\right). \tag{10}$$

Note that there is no one-target distribution.

$$\begin{aligned} 0 &= (\Delta u) - \frac{\tau h}{3a}\left[a\partial_x + b\partial_y\right](\Delta u) \\ &+ \frac{h^2}{3}\left(\partial_{xx}+\partial_{xy}+\partial_{yy}\right)\Delta u + \mathcal{O}(h^3) \end{aligned} \tag{11}$$

2.5 LDA for Laplacian

Viscous fluctuation is the same as in the previous scheme, but distributed by the LDA scheme.

$$\begin{aligned} 0 &= (\Delta u) - \frac{h}{2a}\left[a\partial_x + b\partial_y\right](\Delta u) \\ &+ \frac{h^2}{12a}\left[2a\left(2\partial_{xx}+\partial_{xy}+2\partial_{yy}\right)\Delta u + 3au_{xyyy} - au_{xxyy} + 3b(u_{xxyy}+u_{xxxy})\right] \\ &+\mathcal{O}(h^3) \end{aligned} \tag{12}$$

3 Truncation Error for Advective Part

In this section, "2nd-order" and "3rd-order" refer to the order of accuracy of the fluctuation (2nd-order fluctuation or 3rd-order fluctuation).

3.1 2nd-Order LDA

$$\begin{aligned} 0 &= -(au_x+bu_y) + \frac{h}{2a}\left[a\partial_x + b\partial_y\right](au_x+bu_y) \\ &-\frac{h^2}{6}\left(au_{xxx}+bu_{yyy}+3bu_{xyy}+3bu_{xxy}\right) + \mathcal{O}(h^3) \end{aligned} \tag{13}$$

3.2 3rd-Order LDA

$$\begin{aligned} 0 &= -(au_x+bu_y) + \frac{h}{2a}\left[a\partial_x + b\partial_y\right](au_x+bu_y) \\ &-\frac{h^2}{12a}\left[2a\partial_{xx} + (a+2b)\partial_{xy} + (a+b)\partial_{yy}\right](au_x+bu_y) \\ &+\frac{h^3}{24a}\left[a\partial_{xxx} + b\partial_{yyy} + (a+b)\partial_{xxy} + (a+b)\partial_{xyy}\right](au_x+bu_y) + \mathcal{O}(h^4) \end{aligned}$$

This is not a practical scheme itself, but will be partly used in effect when we combine an advection and a diffusion schemes.

3.3 2nd-Order $1/3$

$$0 \quad = \quad -(au_x + bu_y) - \frac{h^2}{6}\left(\partial_{xx} + \partial_{xy} + \partial_{yy}\right)(au_x + bu_y) + \mathcal{O}(h^4) \tag{14}$$

3.4 3rd-Order $1/3$

$$0 \quad = \quad -(au_x + bu_y) - \frac{5h^2}{36}\left(\partial_{xx} + \partial_{xy} + \partial_{yy}\right)(au_x + bu_y) + \mathcal{O}(h^4) \tag{15}$$

3.5 2nd-Order $SUPG$

$$\begin{aligned}
0 \quad = \quad & -(au_x + bu_y) + \frac{\tau h}{3a}\left(a\partial_x + b\partial_y\right)(au_x + bu_y) \\
& -\frac{h^2}{6}\left(\partial_{xx} + \partial_{xy} + \partial_{yy}\right)(au_x + bu_y) \\
& +\frac{\tau h^3}{36a}\left(6ab\, u_{xxyy} + 4ab\, u_{xxxy} + a^2\, u_{xxxx} + 4ab\, u_{xyyy} + b^2\, u_{yyyy}\right) \\
& +\mathcal{O}(h^4)
\end{aligned} \tag{16}$$

3.6 3rd-Order $SUPG$

$$\begin{aligned}
0 \quad = \quad & -(au_x + bu_y) + \frac{\tau h}{3a}\left(a\partial_x + b\partial_y\right)(au_x + bu_y) \\
& -\frac{5h^2}{36}\left(\partial_{xx} + \partial_{xy} + \partial_{yy}\right)(au_x + bu_y) \\
& +\frac{\tau h^3}{36a}\left[a\partial_{xxx} + b\partial_{yyy} + (a+b)\partial_{xxy} + (a+b)\partial_{xyy}\right](au_x + bu_y) \\
& +\mathcal{O}(h^4)
\end{aligned} \tag{17}$$

4 Truncation Error for Advection-Diffusion Equation

If we combine a viscous scheme and an advection scheme as a simple sum, the truncation error is also a simple sum. Then we immediately find that the resulting scheme is not as accurate as the separate ones.

For example, the sum of the 2nd-order schemes (Galerkin + LDA) gives

$$
\begin{aligned}
0 &= [-(au_x + bu_y) + \nu\,\Delta u] + \frac{h}{2a}\,[a\partial_x + b\partial_y]\,(au_x + bu_y) \\
&\quad -\frac{h^2}{6}\,(au_{xxx} + bu_{yyy} + 3bu_{xyy} + 3bu_{xxy}) + \nu\frac{h^2}{12}\,[u_{xxxx} + u_{yyyy}] \\
&\quad +\mathcal{O}(h^3) \qquad (18)
\end{aligned}
$$

If we use 3rd-order GR scheme for diffusion and the 3rd-order LDA for the advective part as a simple sum, we obtain

$$
\begin{aligned}
0 &= [-(au_x + bu_y) + \nu\,\Delta u] + \frac{h}{2a}\,[a\partial_x + b\partial_y]\,(au_x + bu_y) \\
&\quad -\frac{h^2}{12a}\,[2a\partial_{xx} + (a+2b)\partial_{xy} + (a+b)\partial_{yy}]\,(au_x + bu_y) + \nu\frac{h^2}{12}\,[\partial_{xx} + \partial_{xy} + \partial_{yy}]\,(\Delta u) \\
&\quad +\frac{h^3}{24a}\,[a\partial_{xxx} + b\partial_{yyy} + (a+b)\partial_{xxy} + (a+b)\partial_{xyy}]\,(au_x + bu_y) \\
&\quad +\nu h^4\left[\frac{1}{360}\partial_{xxxx} + \frac{1}{48}\partial_{xxxy} + \frac{43}{720}\partial_{xxyy} + \frac{1}{48}\partial_{xyyy} + \frac{1}{360}\partial_{yyyy}\right](\Delta u) \\
&\quad +\nu\frac{h^4}{24}u_{xxxyyy} + \mathcal{O}(h^4)
\end{aligned}
$$

Alternative approach is to distribute the sum of the fluctuation by a combined distribution coefficient. Combination of the 3rd order schemes (the fluctuation is consistent with the quadratic element).

$$
S_j u_j^{n+1} = S_j u_j^n - \sum_{T\in\{T_j\}} (\beta_{LDA} + \nu/3)\left(\phi^T + \phi_v^T\right) \qquad (19)
$$

$$
\begin{aligned}
0 &= (1+\nu)\,[-(au_x + bu_y) + \nu\,\Delta u] \\
&\quad +\frac{h}{2a}(a\partial_x + b\partial_y)\,[-(au_x + bu_y) + \nu\,\Delta u] + \mathcal{O}(h^2)
\end{aligned}
$$

SUPG can also be used.

$$
S_j u_j^{n+1} = S_j u_j^n - \sum_{T\in\{T_j\}} \frac{1}{3}\left(1 + \tau\frac{k_i}{\sum\max(0,k_i)}\right)\left(\phi^T + \phi_v^T\right) \qquad (20)
$$

Using 3rd-order fluctuation for the advective part, we get

$$
\begin{aligned}
0 &= [-(au_x + bu_y) + \nu\,\Delta u] \\
&\quad -\frac{\tau h}{3a}(a\partial_x + b\partial_y)\,[-(au_x + bu_y) + \nu\,\Delta u] \\
&\quad -\frac{5h^2}{36}\,(\partial_{xx} + \partial_{xy} + \partial_{yy})\,(au_x + bu_y) + \nu\frac{h^2}{3}\,(\partial_{xx} + \partial_{xy} + \partial_{yy})\,\Delta u \\
&\quad +\mathcal{O}(h^3). \qquad (21)
\end{aligned}
$$

Using th 2nd-order fluctuation, we get

$$
\begin{aligned}
0 \;=\; & \left[-(au_x + bu_y) + \nu\,\Delta u\right] \\
& -\frac{\tau h}{3a}(a\partial_x + b\partial_y)\left[-(au_x + bu_y) + \nu\,\Delta u\right] \\
& -\frac{h^2}{6}\left(\partial_{xx} + \partial_{xy} + \partial_{yy}\right)(au_x + bu_y) + \nu\frac{h^2}{3}\left(\partial_{xx} + \partial_{xy} + \partial_{yy}\right)\Delta u \\
& +\mathcal{O}(h^3). \qquad (22)
\end{aligned}
$$

