Change of Variables that Preserves Accuracy

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November 24, 2004

1 Polynomial Representation in DG

Consider a set of variables $\mathbf{U}$ represented within a one-dimensional computational cell ($-\Delta x/2 \leq x \leq \Delta x/2$) as a polynomial of degree n defined by

$$\mathbf{U} = \sum_{i=0}^{n} \overline{\mathbf{V}}^{(i)} \psi_i(x) \tag{1}$$

where $\psi_i(x)$ denotes the Legendre polynomial of degree i,

$$\psi_0(x) = 1, \;\; \psi_1(x) = \frac{x}{\Delta x}, \;\; \psi_2(x) = \left(\frac{x}{\Delta x}\right)^2 - \frac{1}{12}, \tag{2}$$

etc. It is convenient to rewrite this as

$$\mathbf{U} = \sum_{i=0}^{n} \overline{\mathbf{U}}^{(i)} \phi_i(x) \tag{3}$$

where

$$\phi_0(x) = 1, \;\; \phi_1(x) = x, \;\; \phi_2(x) = x^2 - \frac{\Delta x^2}{12}, \tag{4}$$

so that $\overline{\mathbf{U}}^{(i)} = \mathcal{O}(1)$ while $\overline{\mathbf{V}}^{(i)} = \mathcal{O}(\Delta x^i)$. Note also that $x = \mathcal{O}(\Delta x)$ since we only consider the neighborhood of the cell.

Our objective is now to perform the change of variables, say $\mathbf{W}(\mathbf{U})$, without losing accuracy of the representation. We assume that the form of $\mathbf{W}(\mathbf{U})$ and its transformation Jacobian matrix $\frac{\partial \mathbf{W}}{\partial \mathbf{U}}$ are known.

2 Linear Representation (P_1)

Suppose we have a linear representation,

$$\mathbf{U} = \overline{\mathbf{U}}^{(0)} + \overline{\mathbf{U}}^{(1)} x \tag{5}$$

whose error term is clearly $\mathcal{O}(\Delta x^2)$. We wish to represent a set of new variables $\mathbf{W}(\mathbf{U})$ in the same linear form as above. It is easy to show that this can be done within the discretization error: $\mathcal{O}(\Delta x^2)$ in this case. We simply expand $\mathbf{W}(\mathbf{U})$ in x around $x = 0$ which is small whithin a cell,

$$\mathbf{W}\left(\overline{\mathbf{U}}^{(0)} + \overline{\mathbf{U}}^{(1)} x\right) = \mathbf{W}\left(\overline{\mathbf{U}}^{(0)}\right) + \left.\frac{\partial \mathbf{W}}{\partial \mathbf{U}}\right|_{\overline{\mathbf{U}}^{(0)}} \overline{\mathbf{U}}^{(1)} x + \mathcal{O}(\Delta x^2). \tag{6}$$

This shows that the linear representation of $\mathbf{W}$ defined by

$$\mathbf{W} = \overline{\mathbf{W}}^{(0)} + \overline{\mathbf{W}}^{(1)} x \tag{7}$$

where

$$\overline{\mathbf{W}}^{(0)} = \mathbf{W}\left(\overline{\mathbf{U}}^{(0)}\right) \tag{8}$$

$$\overline{\mathbf{W}}^{(1)} = \left.\frac{\partial \mathbf{W}}{\partial \mathbf{U}}\right|_{\overline{\mathbf{U}}^{(0)}} \overline{\mathbf{U}}^{(1)} \tag{9}$$

is accurate up to the error of the original variables, i.e., $\mathcal{O}(\Delta x^2)$.

3 Quadratic Representation (P_2)

Suppose we have a quadratic representation,

$$\mathbf{U} = \overline{\mathbf{U}}^{(0)} + \overline{\mathbf{U}}^{(1)} x + \overline{\mathbf{U}}^{(2)} \left(x^2 - \frac{\Delta x^2}{12}\right) \tag{10}$$

whose error term is clearly $\mathcal{O}(\Delta x^3)$. We then expand in x to get

$$\begin{aligned}
&\mathbf{W}\left(\overline{\mathbf{U}}^{(0)} + \overline{\mathbf{U}}^{(1)} x + \overline{\mathbf{U}}^{(2)} \left(x^2 - \frac{\Delta x^2}{12}\right)\right) = \\
&\mathbf{W}\left(\overline{\mathbf{U}}^{(0)}\right) + \left.\frac{\partial \mathbf{W}}{\partial \mathbf{U}}\right|_{\overline{\mathbf{U}}^{(0)}} \left(\overline{\mathbf{U}}^{(1)} x + \overline{\mathbf{U}}^{(2)} \left(x^2 - \frac{\Delta x^2}{12}\right)\right) \\
&+ \frac{1}{2} \left.\frac{\partial^2 \mathbf{W}}{\partial \mathbf{U}^2}\right|_{\overline{\mathbf{U}}^{(0)}} \cdot \left(\overline{\mathbf{U}}^{(1)} \otimes \overline{\mathbf{U}}^{(1)}\right) x^2 + \mathcal{O}(\Delta x^3).
\end{aligned} \tag{11}$$

Therefore, we have

$$\mathbf{W} = \overline{\mathbf{W}}^{(0)} + \overline{\mathbf{W}}^{(1)} x + \overline{\mathbf{W}}^{(2)} \left(x^2 - \frac{\Delta x^2}{12}\right) + \overline{\mathbf{W}}_p^{(2)} x^2 \tag{12}$$

where

$$\overline{\mathbf{W}}^{(0)} = \mathbf{W}\left(\overline{\mathbf{U}}^{(0)}\right) \tag{13}$$

$$\overline{\mathbf{W}}^{(1)} = \left.\frac{\partial \mathbf{W}}{\partial \mathbf{U}}\right|_{\overline{\mathbf{U}}^{(0)}} \overline{\mathbf{U}}^{(1)} \tag{14}$$

$$\overline{\mathbf{W}}^{(2)} = \left.\frac{\partial \mathbf{W}}{\partial \mathbf{U}}\right|_{\overline{\mathbf{U}}^{(0)}} \overline{\mathbf{U}}^{(2)} \tag{15}$$

$$\overline{\mathbf{W}}_p^{(2)} = \frac{1}{2} \left.\frac{\partial^2 \mathbf{W}}{\partial \mathbf{U}^2}\right|_{\overline{\mathbf{U}}^{(0)}} \cdot \left(\overline{\mathbf{U}}^{(1)} \otimes \overline{\mathbf{U}}^{(1)}\right) \tag{16}$$

Clearly, $\mathbf{W}$ cannot be represented in exactly the same form as $\mathbf{U}$ because of the extra term $\overline{\mathbf{W}}_p^{(2)} x^2$ which is necessary to keep the error within $\mathcal{O}(\Delta x^3)$. It is a standard practice to ignore such extra terms and use a single matrix such as $\frac{\partial \mathbf{W}}{\partial \mathbf{U}}\big|_{\overline{\mathbf{U}}^{(0)}}$ to transform any of high-order moments as in (14) and (15). It is only 2nd-order accurate, but seems to work quite successfully in practice. The reason is simple. It is because we often utilize only a limited amount of information of the transformed variables. For example, if the transformation is used to limit the moments (e.g. limit characteristic variales rather than conservative variables), whatever we do, we change only that moment and therefore we can keep the accuracy of the original variables. On the other hand, if we perform a time integration in the new variables (something like Hancock's predictor-corrector method), then the 2nd-order error may have a significant impact.

