A Triangular Grid Generation by Cauchy-Riemann Equations

Hiroaki Nishikawa*

W.M.Keck Foundation Laboratory for Computational Fluid Dynamics
Department of Aerospace Engineering, University of Michigan, Ann Arbor, Michigan 48109

30th March 1999

Abstract

In this paper, we present a method that generates a smooth triangular grid around an object. A grid in a two-dimensional physical space will be generated by solving Cauchy-Riemann equations in the physical space by treating the grid as unknown. The method is in principle the same as the elliptic grid generation method.

1 Introduction

The grid generation by elliptic partial differential equations has been a popular technique for quite sometime. The main advantage of the method is the fact that the resulting grid is smooth and orthogonal which are very important to the accuracy of the numerical solution. The idea can be easily understood by considering two Laplace's equations with Dirichlet boundary conditions. The contours of the solutions are smooth and nonintersecting, i.e. form a good grid. However, it is in reality impossible to solve the equations in the domain of interest simply because a fixed computational grid is not available in the domain. Therefore we interchange the roles of the independent and dependent variables and solve the transformed equations in the solution space. The resulting equations are highly nonlinear, and therefore they are always solved in an iterative manner starting with a reasonable initial grid. It is very important here to note that the equations we want to solve are in fact Laplace equations in the physical domain.

The method we propose in this paper is based on the fact that two Laplace's equations are equivalent to Cauchy-Riemann equations. That is to say,

$$\xi_{xx} + \xi_{yy} = 0 \quad (1)$$
$$\eta_{xx} + \eta_{yy} = 0 \quad in\ \Omega \quad (2)$$

are equivalent to

$$\xi_x + \eta_y = 0 \quad (3)$$
$$\eta_x - \xi_y = 0 \quad in\ \Omega \quad (4)$$

where Ω is the domain of interest in $x - y$ space, and Dirichlet boundary conditions are assumued to be given for ξ and η on $\partial\Omega$. Now exchanging the variables, we obtain from (1) and (2)

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = 0 \quad (5)$$
$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = 0 \quad (6)$$

*Doctoral Candidate, Aerospace Engineering and Scientific Computing where

$$\alpha = x_\eta^2 + y_\eta^2, \quad \beta = x_\xi x_\eta + y_\xi y_\eta, \quad \gamma = x_\xi^2 + y_\xi^2, \tag{7}$$

and from (5) and (6)

$$x_\xi + y_\eta = 0 \tag{8}$$

$$x_\eta - y_\xi = 0. \tag{9}$$

Here it is important to note that these equations are obtained finally by multiplying(or dividing) both sides by the Jacobian,

$$j = \xi_x \eta_y - \xi_y \eta_x = (y_\eta x_\xi - x_\eta y_\xi)^{-1}. \tag{10}$$

Seemingly the transformed equations of Cauchy-Riemann system (5) and (6) are linear and thus easy to solve while those of Laplace's equations are highly nonlinear and complicated to solve. However the point we like to emphasize is not the linearity but its simpleness. As a matter of fact, the equations that we solve will be nonlinear as shall be described later.

In the next section, we describe the least-squares method for Cauchy-Riemann equations and its use for grid generation. A computational result is presented in the following section.

2 Least-squares Method for CR equations

We begin by considering CR equations in the physical domain, which are the equations we want to solve as mentioned earlier. Let us first suppose that we have triangulated the physical domain on which CR equations can be solved, and denote the set of these triangles by $\{T\}$ and the set of vertices by $\{V\} = \{V_{int}\} \cup \{V_b\}$ where $\{V_{int}\}$ is the set of interior vertices and $\{V_b\}$ is the set of boundary vertices. We then store the solution values at vertices and assume that the solutions vary linearly within each element. We now obtain the residuals by integrating CR equations within each element,

$$U_T = \iint_T (\xi_x + \eta_y)\, dxdy = \frac{1}{2}\sum_{i \in j_T} \xi_i \Delta y_i - \frac{1}{2}\sum_{i \in j_T} \eta_i \Delta x_i \tag{11}$$

$$V_T = \iint_T (\xi_y - \eta_x)\, dxdy = \frac{1}{2}\sum_{i \in j_T} \xi_i \Delta x_i + \frac{1}{2}\sum_{i \in j_T} \eta_i \Delta y_i \tag{12}$$

where j_T is a set of vertices that defines the triangle T and $\Delta\{\}_i$ denotes a difference taken counterclockwise along the side opposite to i. The nodal solutions are obtained by minimizing the least-squares norm defined by

$$\mathcal{F} = \sum_{T \in \{T\}} F_T = \sum_{T \in \{T\}} \frac{1}{2S_T}\left[{U_T}^2 + {V_T}^2\right] \quad . \tag{13}$$

where S_T is the area of the element given by

$$S_T = \frac{1}{2}\sum_{i \in j_T} x_i \Delta y_i = -\frac{1}{2}\sum_{i \in j_T} y_i \Delta x_i. \tag{14}$$

It has been known that this norm can be rewritten as

$$\mathcal{F} = \sum_{T \in \{T\}} \frac{1}{2}\iint_T \nabla\xi \cdot \nabla\xi\, dxdy \;+ \sum_{T \in \{T\}} \frac{1}{2}\iint_T \nabla\eta \cdot \nabla\eta\, dxdy \;+ \sum_{T \in \{T_h\}} S_T \tag{15}$$

where $\{T_h\}$ is the image in $\xi - \eta$ space of the triangulation $\{T\}$, and J_T is the discrete version of the Jacobian in element T. Note that the last term is irrelevant to the minimization since it involves only the boundary values of ξ and η which are given as boundary conditions. Now we see clearly that (15) is nothing but the energy norm for the two Laplace's equations, and thus the solutions are in effect equivalent to finite-element solutions. Even more importantly, it is understood that minimizing this norm is equivalent to solving the two Laplace's equations by finite-element method. Recall again that the equations we want to solve are the Laplace's equaitons and also that we want to generate a grid in $x-y$ space. We then propose to switch the minimization variables, i.e. minimize the energy norm with respect to not (ξ_j, η_j) but (x_j, y_j), $j \in \{V_{int}\}$. Although the norm is not quadratic any more, in this way we can solve the proper equations as well as treat the grid as unknown once a grid in $\xi-\eta$ space and the boundary conditions are chosen by an appropriate topology of the final grid, e.g. O-grid. Therefore the equations we need to solve are

$$\frac{\partial \mathcal{F}}{\partial x_j} = \sum_{T \in \{T_j\}} \frac{\partial F_T}{\partial x_j} = 0 \ \ \forall j \in \{V_{int}\} \tag{16}$$

$$\frac{\partial \mathcal{F}}{\partial y_j} = \sum_{T \in \{T_j\}} \frac{\partial F_T}{\partial y_j} = 0 \ \ \forall j \in \{V_{int}\}. \tag{17}$$

where $\{T_j\}$ is a group of triangles that share the vertex j. Straightforward differentiation yields

$$\sum_{T \in \{T_j\}} \frac{1}{2S_T} [\ \ \Delta\eta_T U_T - \Delta\xi_T V_T - F_T \Delta y_T] = 0 \ \ \forall j \in \{V_{int}\} \tag{18}$$

$$\sum_{T \in \{T_j\}} \frac{1}{2S_T} [-\Delta\eta_T V_T - \Delta\xi_T U_T + F_T \Delta x_T] = 0 \ \ \forall j \in \{V_{int}\} \tag{19}$$

where $\Delta\{\}_T$ denotes a difference taken counterclockwise along the side of triangle T . These are nonlinear equations for (x_j, y_j), and therefore must be solved by an iterative method.

3 Grid Controls

To control the quality of the grid such as stretching or orthogonality, it is necessary to introduce source terms in the Cauchy-Riemann equations. We begin by writing the equations in the following form.

$$\xi_x + \eta_y = j\lambda \tag{20}$$

$$\eta_x - \xi_y = j\omega \quad in\ \Omega \tag{21}$$

Then, assuming λ and ω vary linearly withn each element, we have

$$U_T = \frac{1}{2} \sum_{i \in j_T} \xi_i \Delta y_i - \frac{1}{2} \sum_{i \in j_T} \eta_i \Delta x_i - \bar{\lambda}_T S_{Th} \tag{22}$$

$$V_T = \frac{1}{2} \sum_{i \in j_T} \xi_i \Delta x_i + \frac{1}{2} \sum_{i \in j_T} \eta_i \Delta y_i - \bar{\omega}_T S_{Th} \tag{23}$$

4 Implementation

4.1 Iterative method

The problem that we have being a nonquadratic minimization problem, the simplest method would be a method of steepest descent,

$$x_j^{n+1} = x_j^n - c_x \frac{\partial \mathcal{F}}{\partial x_j}, \quad y_j^{n+1} = y_j^n - c_y \frac{\partial \mathcal{F}}{\partial y_j} \tag{24}$$

where c_x and c_y are small constants. The steepest descent method is, however, well-known to be very slow. One reason for the poor performance is its dimensional inconsistency, e.g. $\frac{\partial \mathcal{F}}{\partial x_j}$ does not have the dimension of x. One possible way to rescue this situation is the Newton's method which uses the inverse of the Hessian matrix. In this study, however for simplicity, we use only the diagonal part of the Hessian which is easy to invert and still achieves the dimensional consistency. Hence the constant c is replaced by

$$c_x = \omega \left(\frac{\partial^2 \mathcal{F}}{\partial x_j^2} \right)^{-1}, \quad c_y = \omega \left(\frac{\partial^2 \mathcal{F}}{\partial y_j^2} \right)^{-1} \tag{25}$$

where ω is a small constant and the second derivatives are given by

$$\frac{\partial^2 \mathcal{F}}{\partial x_j^2} = \sum_{T \in \{T_j\}} \frac{1}{S_T} \left[\frac{1}{4} \{ (\Delta \xi_T)^2 + (\Delta \eta_T)^2 \} - \frac{\partial F_T}{\partial x_j} \Delta y_T \right] \tag{26}$$

$$\frac{\partial^2 \mathcal{F}}{\partial y_j^2} = \sum_{T \in \{T_j\}} \frac{1}{S_T} \left[\frac{1}{4} \{ (\Delta \xi_T)^2 + (\Delta \eta_T)^2 \} + \frac{\partial F_T}{\partial y_j} \Delta x_T \right]. \tag{27}$$

Another acceleration technique that we use is an iteration of Gauss-Seidel type where (x_j, y_j) are updated successively and thus the most recently calculated nodal values are used for the updates. These are in fact found, from numerical experiments, to be very effective for our problem.

4.2 Diagonal swapping

The minimization strategy described so far preserves the mesh topology. This may be considered as a defect since the initial topology might not be satisfactory especially if an irregularity exits in the the domain of interest, e.g. trailing edge of an airfoil. The diagonal swapping can be used to alleviate the restriction. This is a well-known technique to construct Delaunay triangulations, which replaces the existing diagonals in all the possible quadrilaterals in the mesh whenever it leads to shorter diagonals. A suitable criterion for the minimization problem would be to swap the diagonals whenever it yields smaller norm. In practice we first search the candidate edges by comparing the sum of two F_T's of the triangles that share an edge with that of the other possible pair of the triangles. Next we put them in order of large reduction of the norm and start the swappings. However every time we swap one diagonal, the remaining edges to be swapped may not yield smaller norms anymore. Therefore we check the effectiveness of the swapping again every time before an edge is actually swapped.

5 Numerical experiments

We have generated O-grid and C-grid around a von Karman-Trefftz airfoil by applying the Cauchy-Riemann method. Here diagonal swapping was not used, instead a favorable mesh topology was chosen initially. In both cases, the initial grids were generated algebraically. The results are shown in the following figures. As can be seen in Figure 3, the C-grid generated by the Cauchy-Riemann equations is not very good in the sense that the grid is not orthogonal near the airfoil surface and that the spacing near the trailing edge and the wake region is large. In fact the initial grid generated algebraically has better features. In order to control the qualities of the grid, it may be necessary to introduce source terms in the Cauchy-Riemann equations, i.e. source or vorticity.

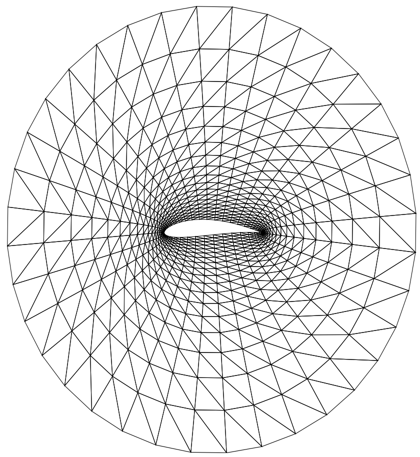

Figure 1: A O-grid generated by the Cauchy-Riemann equations

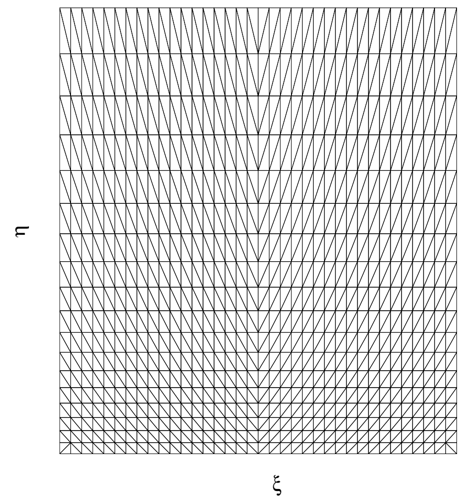

Figure 2: The grid in the solution space.

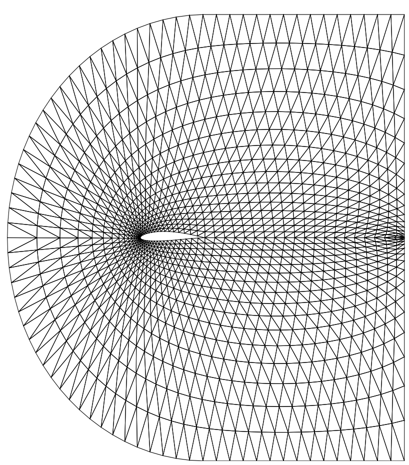

Figure 3: A C-grid generated by the Cauchy-Riemann equations.

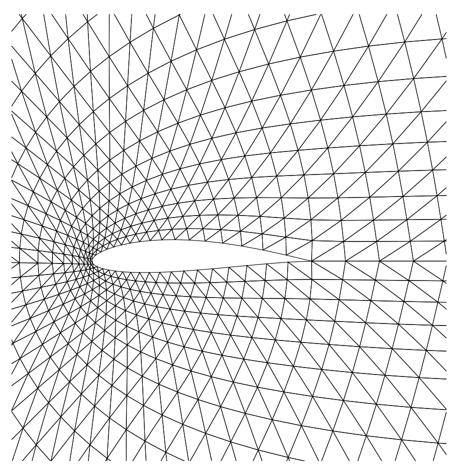

Figure 4: The initial grid constructed algebraically.

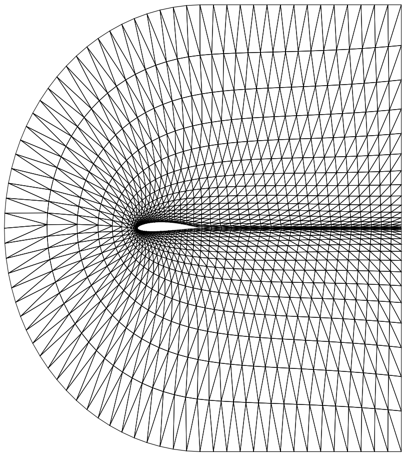

Figure 5: A C-grid generated by the Cauchy-Riemann equations with source terms.

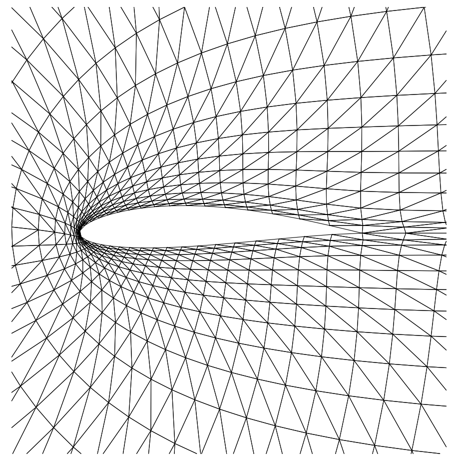

Figure 6: A close view around the airfoil.

6 Concluding Remarks

It was shown that Cauchy-Riemann equations can be used to generate triangular grids around an object. In order to make the method more practical, it is necessary to develop a technique by which we can control the quality of the grid. It appears that this can be done by introducing source terms in the Cauchy-Riemann equations.

References

Steger, J. L. and Sorenson, R.L., Automatic mesh-point clustering near a boundary in grid generation with elliptic partial differential equations, Journal of Computational Physics, **33**, pp.405-410.