

Norm-Reducing Schemes for Linear Advection on Triangular Grids

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1 Norm-Reducing Property

We consider solving the linear advection equation

$$u_t + a\partial_x u + b\partial_y u = 0 \quad (1)$$

in a triangulated domain. Define the fluctuation over the triangle T ,

$$\phi = \int_T \mathbf{u}_t = \int_T (a\partial_x u + b\partial_y u) \quad (2)$$

Storing the solutions at vertices, and assuming a piecewise linear variation over the triangle with vertices $i = 1, 2, 3$, we obtain

$$\phi = \sum_i k_i u_i \quad (3)$$

where $k_i = \frac{1}{2}(a, b) \cdot \mathbf{n}_i$ and \mathbf{n}_i is the inward normal vector of the edge opposite to the node j . Note that the fluctuation ϕ can be rewritten in many different ways:

$$\phi = k_1(u_1 - u_3) + k_2(u_2 - u_3) \quad (4)$$

$$= \sum_i k_i^+ (u_{out} - u_{in}) \quad (5)$$

where $u_{out} = \sum_i k_i^+ u_i / \sum k_i^+$, $u_{in} = \sum_i k_i^- u_i / \sum k_i^-$, $k_i^+ = \max(0, k_i)$, and $k_i^- = \min(0, k_i)$. To update the nodal solutions, we distribute the fluctuation to the nodes in the following form,

$$u_j^{n+1} \leftarrow u_j^n - \omega \phi_i \quad (6)$$

where ω is a positive constant and thus ϕ_i is a fraction of the fluctuation to be distributed to the node j , i.e. $\sum_i \phi_i = \phi$. Various schemes are available to split the fluctuation into appropriate fractions, such as the N scheme, the PSI scheme, the LDA scheme, etc. These schemes are classified by some useful properties such as positivity and linearity-preserving. In this paper, we introduce a yet another property that we call *norm-reducing*. Based on the observation that no schemes can drive all the

fluctuations to zero at convergence on triangular grids, we ask if a scheme is reducing the fluctuations. To see this, consider the L_2 norm

$$\mathcal{F} = \frac{1}{2} \sum_T \phi^T Q^T \phi^T \quad (7)$$

where Q^T is a positive weight. Its variation due to a small change in the solution at node j , δu_j , is given by

$$\delta \mathcal{F} = \sum_{\{T_j\}} \delta \mathcal{F}^T = \sum_{\{T_j\}} k_j^T \phi^T \delta u_j \quad (8)$$

where $\{T_j\}$ is a group of triangles that share the node j , and we have taken $Q^T = 1$ for simplicity. If the small change δu_j is caused by a distribution scheme in the form (6), this can be written as

$$\delta \mathcal{F} = -\omega \sum_{\{T_j\}} k_j^T \phi^T \phi_j^T. \quad (9)$$

Then, the scheme is norm-reducing if

$$\sum_{\{T_j\}} k_j^T \phi^T \phi_j^T > 0 \quad (10)$$

which guarantees that

$$\delta \mathcal{F} < 0. \quad (11)$$

As in the positivity property, it is often easier to consider its local counterpart although this typically yields stronger conditions. Let us consider a local contribution from triangle $T \in \{T_j\}$, dropping the superscript T ,

$$\delta \mathcal{F}^T = -\omega k_j \phi \phi_j. \quad (12)$$

A scheme is then called locally norm-reducing within a triangle T if

$$k_j \phi \phi_j > 0 \quad (13)$$

which guarantees that

$$\delta \mathcal{F}^T < 0. \quad (14)$$

2 Norm-Reducing Schemes

It is easy to prove that *sending the whole fluctuation to the downwind node guarantees norm-reducing*. This is simply because in such a case $\phi_j = \phi$ for $k_j > 0$, i.e.

$$k_j \phi \phi_j = k_j (\phi)^2 > 0 \quad (15)$$

and thus norm-reducing. Note that a scheme is norm-reducing whenever it sends the whole fluctuation to the node for which $k_i > 0$, and this includes upwind schemes in the one-target situation. It is interesting that this shows that sending the whole

fluctuation to upwind nodes for which $k_j < 0$ yields a norm-increasing scheme, thus implying instability.

From here on, we consider the two-target case only and assume that $k_1 > 0$, $k_2 > 0$, and $k_3 < 0$ and that $\phi_3 = 0$. We then consider the variation of the norm due to δu_1 and δu_2 ,

$$\delta \mathcal{F}^T = -\omega k_1 \phi \phi_1 - \omega k_2 \phi \phi_2 = -\omega \phi (k_1 \phi_1 + k_2 \phi_2) \quad (16)$$

which can be expanded, using

$$\phi_1 + \phi_2 = \phi, \quad (17)$$

as

$$\delta \mathcal{F}^T = -\omega (\phi_1 + \phi_2) (k_1 \phi_1 + k_2 \phi_2) \quad (18)$$

$$= -\omega \{k_1 \phi_1^2 + k_2 \phi_2^2 + (k_1 + k_2) \phi_1 \phi_2\} \quad (19)$$

$$= -\omega \left(\sqrt{k_1} \phi_1 + \sqrt{k_2} \phi_2 \right)^2 - \omega \left(\sqrt{k_1} - \sqrt{k_2} \right)^2 \phi_1 \phi_2. \quad (20)$$

It is obvious from this that the norm may increase if $\phi_1 \phi_2$ is negative. Also note that a scheme is norm-reducing if $k_1 = k_2$.

The condition for norm-reducing can be expressed in various forms. An example is

$$k_1 \phi_1^2 + k_2 \phi_2^2 + (k_1 + k_2) \phi_1 \phi_2 > 0 \quad (21)$$

or

$$\phi_1 \phi_2 > -\frac{k_1 \phi_1^2 + k_2 \phi_2^2}{k_1 + k_2} \quad (22)$$

This means that the product $\phi_1 \phi_2$ does not have to be positive but must be greater than a certain negative value.

We first consider the LDA scheme, which is in the two-target case given by

$$\phi_1 = \frac{k_1}{k_1 + k_2} \phi \quad (23)$$

$$\phi_2 = \frac{k_2}{k_1 + k_2} \phi. \quad (24)$$

Substituting these into (16) gives

$$\delta \mathcal{F}^T = -\omega \phi^2 \frac{k_1^2 + k_2^2}{k_1 + k_2} < 0 \quad (25)$$

Therefore, the LDA scheme is norm-reducing. This is an obvious result because the LDA scheme may be obtained by applying an upwind constraint on the least-squares minimization scheme. The N scheme is given by

$$\phi_1 = k_1 (u_1 - u_3) \quad (26)$$

$$\phi_2 = k_2 (u_2 - u_3). \quad (27)$$

The product of these two terms can be negative, and therefore the N scheme is not norm-reducing when it is so. The PSI scheme is a limited-version of the N scheme, and it differs from the N scheme in that it sends the whole fluctuation to the node with $k_i > 0$ and $\phi_i/\phi > 0$ when $\phi_1\phi_2 < 0$. This makes the PSI scheme fully norm-reducing.

The condition for norm-reducing may be expressed in terms of distribution coefficients. From (16), we have

$$\delta\mathcal{F}^T = -\omega\phi^2(k_1\beta_1 + k_2\beta_2) \quad (28)$$

where $\beta_i = \phi_i/\phi$, and therefore a scheme is norm-reducing if

$$k_1\beta_1 + k_2\beta_2 > 0 \quad (29)$$

which can be generalized as

$$\sum_i k_i \beta_i > 0 \quad (30)$$

Consider a scheme in the form,

$$\beta_i = \frac{1}{3} + Ck_i \quad (31)$$

where C is a positive constant. This includes the Lax-Wendroff scheme with

$$C = \frac{\Delta t}{2S_T} \quad (32)$$

and also the SUPG scheme with

$$C = \frac{h}{2|\lambda|S_T}. \quad (33)$$

This type of scheme is norm-reducing because

$$\sum_i k_i \beta_i = \sum_i k_i \left(\frac{1}{3} + Ck_i \right) \quad (34)$$

$$= \frac{1}{3} \sum_i k_i + C \sum_i k_i^2 \quad (35)$$

$$= C \sum_i k_i^2 > 0 \quad (36)$$